

# THE SCOTT TOPOLOGY OF A ROOTED NON-METRIC TREE

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**ABSTRACT.** In this paper we prove that the Scott topology  $\mathfrak{S}$  on a rooted non-metric tree  $\mathcal{T}$  is strictly coarser than the weak tree topology. Moreover, for each  $t \in \mathcal{T}$ , we consider a natural order  $\preceq_t$  on  $\mathcal{T}$  under which  $t$  is the root of  $\mathcal{T}$ . Then the weak tree topology is generated by the union of the Scott topologies  $\mathfrak{S}_t$  associated to  $\preceq_t$ .

## 1. INTRODUCTION

During my presentation at “ALANT 3 – Joint Conferences on Algebra, Logic and Number Theory” in Będlewo, Poland, many questions were raised about the different topologies on a rooted non-metric tree (see Definition 2.1). For instance, is the weak tree topology (see Definition 2.4) the same as the Scott topology (see Definition 3.1)? This paper serves to answer those questions.

Favre and Jonsson introduced the valuative tree in [1]. They considered valuations centered at the ring  $\mathbb{C}[[x, y]]$  of the formal Laurent series in two variables over the field of complex numbers. In order to axiomatize some properties of this object, they introduced the concept of a rooted non-metric tree. In [2], Granja studied the equivalent case for valuations centered at a fixed two-dimensional regular local ring. In both works, the definition of rooted non-metric tree is not satisfactory (see discussion about that in [3]). In [3], we complete this definition and compare some natural topologies on a rooted non-metric tree.

Since a rooted non-metric tree  $(\mathcal{T}, \preceq)$  is, by definition, a partially ordered set we can consider the Scott topology on it. In this paper, we prove the following:

**Theorem 1.1.** *The Scott topology on  $\mathcal{T}$  is strictly coarser than the weak tree topology.*

For each point  $t \in \mathcal{T}$ , we can define an order  $\preceq_t$  on  $\mathcal{T}$ , such that  $(\mathcal{T}, \preceq)$  and  $(\mathcal{T}, \preceq_t)$  have the same segments (see Definition 2.4) and under which  $t$  is a root of  $\mathcal{T}$ . Theorem 1.1 is a consequence of the following stronger result:

**Theorem 1.2.** *For each  $t \in \mathcal{T}$ , consider the Scott topology  $\mathfrak{S}_t$  on  $\mathcal{T}$  associated to  $\preceq_t$ . Then the weak tree topology on  $\mathcal{T}$  is the topology generated by  $\bigcup_{t \in \mathcal{T}} \mathfrak{S}_t$ .*

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## 2. THE VALUATIVE TREE

**Definition 2.1.** A **rooted non-metric tree** is a partially ordered set  $(\mathcal{T}, \preceq)$  such that:

- (T1): There exists a (unique) smallest element  $t_0 \in \mathcal{T}$ .
- (T2): Every set of the form  $I_t = \{a \in \mathcal{T} \mid a \preceq t\}$  is isomorphic (as ordered sets) to a real interval.
- (T3): Every totally ordered convex subset of  $\mathcal{T}$  is isomorphic to a real interval.
- (T4): Every non-empty subset  $\mathcal{S}$  of  $\mathcal{T}$  admits an infimum in  $\mathcal{T}$ .

In [3], we prove the following:

**Lemma 2.2.** *Under conditions (T1) and (T2), the following conditions are equivalent:*

- (T4): *Every non-empty subset  $\mathcal{S} \subseteq \mathcal{T}$  admits an infimum.*
- (T4'): *Given two elements  $a, b \in \mathcal{T}$ , the set  $\{a, b\}$  admits an infimum  $a \wedge b$ .*

**Remark 2.3.** The lemma above shows that if a partially ordered set for which conditions (T1) and (T2) hold is directed (with respect to reverse set inclusion), then its order is a directed complete partial order (with respect to reverse set inclusion).

We will now define some properties associated to a rooted non-metric tree.

**Definition 2.4.** (i): Given a non-empty subset  $\mathcal{S} \subseteq \mathcal{T}$  we define the **join**

$\bigwedge_{s \in \mathcal{S}} s$  of  $\mathcal{S}$  to be the infimum of  $\mathcal{S}$ .

(ii): Given two elements  $a, b \in \mathcal{T}$  we define the **closed segment** connecting them by

$$[a, b] := \{c \in \mathcal{T} \mid (a \wedge b \preceq c \preceq a) \vee (a \wedge b \preceq c \preceq b)\}.$$

We define  $]a, b]$  and  $[a, b[$  similarly.

(iii): For  $t \in \mathcal{T}$  we define an equivalence relation on  $\mathcal{T} \setminus \{t\}$  by setting

$$a \sim_t b \iff t \notin [a, b].$$

For an element  $a \in \mathcal{T} \setminus \{t\}$  its equivalence class will be denoted by  $[a]_t$ , i.e.,  $[a]_t = \{b \in \mathcal{T} \mid a \sim_t b\}$ . Denote by  $\mathcal{T}_t$  to the set  $\{[a]_t \mid a \in \mathcal{T}\}$  of all equivalence classes under  $\sim_t$

- (iv): The **weak tree topology** on  $\mathcal{T}$  is the topology generated by all the sets of the form  $[a]_t$  for  $a$  and  $t$  running through  $\mathcal{T}$ .
- (v): A **parametrization** of a rooted non-metric tree is an increasing (or decreasing) mapping  $\Psi : \mathcal{T} \longrightarrow [-\infty, \infty]$  such that its restriction to every totally ordered convex subset of  $\mathcal{T}$  is an isomorphism (of ordered sets) onto a real interval.

(vi): Given an increasing parametrization  $\Psi : \mathcal{T} \rightarrow [1, \infty]$  we define a metric on  $\mathcal{T}$  by setting

$$d_\Psi(a, b) = \left( \frac{1}{\Psi(a \wedge b)} - \frac{1}{\Psi(a)} \right) + \left( \frac{1}{\Psi(a \wedge b)} - \frac{1}{\Psi(b)} \right).$$

In [3], we prove the following two results:

**Theorem 2.5.** *Let  $(\mathcal{T}, \preceq)$  be a rooted non-metric tree and let  $\Psi : \mathcal{T} \rightarrow [1, \infty]$  be a parametrization of  $\mathcal{T}$ . Then the weak tree topology on  $\mathcal{T}$  is coarser than or equal to the topology associated with the metric  $d_\Psi$ .*

**Theorem 2.6.** *If there is an element  $t \in \mathcal{T}$  with uncountably many branches (i.e., if  $|\mathcal{T}_t| > |\mathbb{N}|$ ), then the weak tree topology is not first countable. In particular, the metric topology given by any parametrization is strictly coarser than the weak tree topology.*

We fix an element  $t \in \mathcal{T}$  and define a relation  $\preceq_t$  on  $\mathcal{T}$  as follows. For any two elements  $a, b \in \mathcal{T}$  we declare that  $a \preceq_t b$  if and only if  $a \in [t, b]$ . It is straightforward to prove that  $\preceq_t$  is an order on  $\mathcal{T}$  and that  $(\mathcal{T}, \preceq_t)$  is a rooted non-metric tree (for which  $t$  is a root). Moreover, the segments under this new order are exactly the same as those defined by the order  $\preceq$ . Consequently, the weak tree topology on  $\mathcal{T}$  defined by these two orders is the same.

The following lemma will be used in the proof of the main theorem.

**Lemma 2.7.** *Take elements  $a, b, c$  in a rooted non-metric tree  $\mathcal{T}$ . Then we have  $[a, c] \subseteq [a, b] \cup [b, c]$ .*

*Proof.* We have to prove that both segments  $[a \wedge c, a]$  and  $[a \wedge c, c]$  are subsets of  $[a, b] \cup [b, c]$ .

Consider the segment  $I_b$ , which is totally ordered by property **(T2)**. Since  $b \wedge c \preceq b$  and  $a \wedge b \preceq b$ , these two elements are comparable. If  $a \wedge b \preceq b \wedge c$ , then  $a \wedge b \preceq c$ . Hence,  $a \wedge b \preceq a \wedge c$ , which implies that  $[a \wedge c, a] \subseteq [a \wedge b, a] \subseteq [a, b]$ .

If  $b \wedge c \preceq a \wedge b$ , then  $b \wedge c \preceq a$  and consequently  $b \wedge c \preceq a \wedge c$ . On the other hand, since  $a \wedge b \preceq a$  and  $a \wedge c \preceq a$  we have that  $a \wedge b \preceq a \wedge c$  or  $a \wedge c \preceq a \wedge b$ . In the first case we reason like in the previous paragraph to obtain  $[a \wedge c, a] \subseteq [a, b]$ . If  $a \wedge c \preceq a \wedge b$ , then  $a \wedge c \preceq b$  and consequently  $a \wedge c \preceq b \wedge c$ . Hence,  $a \wedge c = b \wedge c$  and thus

$$[a \wedge c, a \wedge b] = [b \wedge c, a \wedge b] \subseteq [b \wedge c, b] \subseteq [b, c].$$

Therefore

$$[a \wedge c, a] = [a \wedge c, a \wedge b] \cup [a \wedge b, a] \subseteq [b, c] \cup [a, b].$$

The proof that  $[a \wedge c, c] \subseteq [a, b] \cup [b, c]$  is analogous.  $\square$

## 3. THE SCOTT TOPOLOGY

Consider a partially ordered set  $(\mathcal{P}, \preceq)$ . A subset  $\mathcal{S}$  of  $\mathcal{P}$  is said to be an **upper set** if for every elements  $x, y \in \mathcal{P}$ , if  $x \in \mathcal{S}$  and  $y \geq x$ , then  $y \in \mathcal{S}$ . The set  $\mathcal{S}$  is said to be **inaccessible by directed joints** if for every directed subset  $\mathcal{D}$  of  $\mathcal{P}$ , if  $\sup \mathcal{D} \in \mathcal{S}$ , then  $\mathcal{D} \cap \mathcal{S} \neq \emptyset$ .

**Definition 3.1.** The Scott topology  $\mathfrak{S}$  on  $\mathcal{P}$  is defined by setting as open sets the upper sets which are inaccessible by directed joints.

**Proposition 3.2.** *Let  $(\mathcal{T}, \preceq)$  be a rooted non-metric tree. Then every Scott open set  $\mathcal{O}$  of  $\mathcal{T}$  is open in the weak tree topology.*

*Proof.* For each point  $a \in \mathcal{O}$ , we will prove that there exists  $t \in \mathcal{T}$  such that  $[a]_t \subseteq \mathcal{O}$ . Consider the set  $\mathcal{D} := \{t' \in \mathcal{T} \mid t' \preceq a \text{ and } t' \neq a\}$ . By property **(T2)**, this set is order isomorphic to a real interval (thus a directed set). Hence,  $\sup \mathcal{D} = a \in \mathcal{O}$  and since  $\mathcal{O}$  is open in the Scott topology, we have that  $\mathcal{O} \cap \mathcal{D} \neq \emptyset$ .

Take any  $t \in \mathcal{O} \cap \mathcal{D}$ . For each  $b \in [a]_t$ , we will prove that  $b \in \mathcal{O}$ . Suppose, towards a contradiction, that  $t \not\preceq b$ . Since  $t \preceq a$  and  $I_a$  is totally ordered we have  $t \preceq a \wedge b$  or  $a \wedge b \preceq t$ . The first case cannot happen because  $t \not\preceq b$  and  $a \wedge b \preceq b$ . Consequently,  $a \wedge b \preceq t \preceq a$  and hence  $t \in [a, b]$ . This means that  $b \notin [a]_t$ , a contradiction. Hence,  $t \preceq b$ . Since  $t \in \mathcal{O}$  and  $\mathcal{O}$  is an upper set we obtain that  $b \in \mathcal{O}$ . Therefore,  $[a]_t \subseteq \mathcal{O}$ , which concludes our proof.  $\square$

*Proof of Theorem 1.1.* The previous proposition shows that the Scott topology is coarser than the weak tree topology. It remains to show that they are different.

Take  $t \in \mathcal{T}$  such that  $t$  is not the root of  $(\mathcal{T}, \preceq)$ . Consider the open subbasic set  $[t_0]_t$  in the weak tree topology. Then  $t_0 \in [t_0]_t$ ,  $t_0 \preceq t$  but  $t \notin [t_0]_t$  which implies that  $[t_0]_t$  is not an upper set. Hence,  $[t_0]_t$  is not open in the Scott topology.  $\square$

**Remark 3.3.** There are many ways to see that the Scott topology is not the weak tree topology. For instance, the weak tree topology is Hausdorff, but the Scott topology (of a rooted non-metric tree) is not. Also, if we consider the orders  $\preceq_t$  and  $\preceq_s$  on  $\mathcal{T}$  for  $t \neq s$ , then the associated weak tree topologies are the same, but the Scott topologies  $\mathfrak{S}_t$  and  $\mathfrak{S}_s$  are not.

The next result will be useful to prove Theorem 1.2.

**Lemma 3.4.** *For the rooted non-metric tree  $(\mathcal{T}, \preceq)$ , every subbasic open set in the weak tree topology is inaccessible by directed joints.*

*Proof.* Consider a subbasic open set  $[a]_t$  of  $\mathcal{T}$  and take a directed set  $\mathcal{D}$  such that  $\mathcal{D} \subseteq \mathcal{T} \setminus [a]_t$ . We want to prove that  $\sup \mathcal{D} =: b \notin [a]_t$ . If  $\mathcal{D} = \{t\}$ , then  $\sup \mathcal{D} = t \notin [a]_t$  and we are done. Hence, assume that there exists  $d \in \mathcal{D}$  such that  $d \neq t$ .

Assume first that  $t \not\preceq a$ . We will show that  $t \preceq \mathcal{D}$  (i.e.,  $t \preceq d$  for every  $d \in \mathcal{D}$ ). Consequently, also  $t \preceq b$  and thus  $b \notin [a]_t$ . For an element  $d \in \mathcal{D}$ , if  $t \preceq a \wedge d$ , then in particular  $t \preceq a$ , which is a contradiction. Hence,  $t \not\preceq a \wedge d$ . If  $t \not\preceq d$ , then

$t \notin [d, a]$  which is a contradiction to  $\mathcal{D} \subseteq \mathcal{T} \setminus [a]_t$ . Hence,  $t \preceq d$ , which is what we wanted to prove.

Assume now that  $t \preceq a$ . If we prove that  $a \wedge b \preceq t$ , then  $t \in [a, b]$  and this will conclude our proof. Since  $t \preceq a$  and  $a \wedge b \preceq a$ , we have that  $t \prec a \wedge b$  if  $a \wedge b \preceq t$ . Suppose, towards a contradiction, that  $t \prec a \wedge b$ . If  $a \wedge b \prec b$ , then there exists  $d \in \mathcal{D}$  such that  $a \wedge b \prec d \preceq b$ . This implies that  $a \wedge b = a \wedge d$  and consequently  $t \notin [a, d]$ . This is a contradiction to  $d \notin [a]_t$ . If  $a \wedge b = b$ , then  $b \preceq a$ . Since  $b = \sup \mathcal{D}$ , there exists  $d \in \mathcal{D}$  such that  $t \prec d \preceq b \preceq a$  and consequently  $t \notin [a, d]$ , which is again a contradiction.  $\square$

**Lemma 3.5.** *If  $t$  is the root of  $\mathcal{T}$ , then the set  $[a]_t$  is an upper set for every  $a \in \mathcal{T} \setminus \{t\}$ .*

*Proof.* Take elements  $b, c \in \mathcal{T}$  such that  $b \in [a]_t$  and  $b \prec c$ . We will show that  $c \in [a]_t$ . Since  $b \prec c$ , we have  $[b, c] = \{d \in \mathcal{T} \mid b \preceq d \preceq c\}$ . Since  $t$  is the smallest element of  $\mathcal{T}$  we have that  $t \notin [b, c]$ . On the other hand, since  $b \in [a]_t$ , by the definition of  $[a]_t$ , we also have that  $t \notin [a, b]$ . Therefore, we use Lemma 2.7 to conclude that  $t \notin [a, c] \subseteq [a, b] \cup [b, c]$ . Therefore,  $c \in [a]_t$ .  $\square$

We are now ready to prove our main result.

*Proof of Theorem 1.2.* As a consequence of Lemma 3.5, every set of the form  $[a]_t$  is an upper set with respect to the order  $\preceq_t$ . Moreover, applying Lemma 3.4 to  $(\mathcal{T}, \preceq_t)$  we obtain that this set is inaccessible by directed joints. Hence,  $[a]_t \in \mathfrak{S}_t$ .

On the other hand, by Proposition 3.2, every open set in the Scott topology  $\mathcal{S}_t$  is open in the weak tree topology. Therefore, the weak tree topology is generated by  $\bigcup_{t \in \mathcal{T}} \mathfrak{S}_t$ .  $\square$

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